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B. Sc. Part III

Paper — 5

Classical Mechanics

which is Hamilton's Principle.

Example – use Hamilton's Principle to find the eqn of motion of one-dimentional harmonic oscillator.

Sol. The Kinetic energy of harmonic oscillator

$$T = \frac{1}{2} m x^2.$$

The Potential energy of harmonic oscillator

$$V = -\int F dx = \int kx\, dx = \frac{1}{2} k x^2$$

∴ The lagrangian of the system

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Teacher's Signature :

According to the Hamilton's Principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2\right) dt = 0 \quad \text{--- (1)}$$

or $$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2\right) dt = 0$$

or $$\int_{t_1}^{t_2} \delta \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2\right) dt = 0$$

or $$\int_{t_1}^{t_2} (m \dot{x}\, \delta \dot{x} - K x\, \delta x)\, dt = 0$$

But $\delta \dot{x} = \frac{d}{dt}(\delta x)$

or $$\int_{t_1}^{t_2} m \dot{x} \frac{d}{dt}(\delta x)\, dt - \int_{t_1}^{t_2} K x\, \delta x\, dt = 0$$

or $$\left[m \dot{x}\, \delta x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x})\, \delta x\, dt - \int_{t_1}^{t_2} K x\, \delta x\, dt = 0 \quad \text{--- (2)}$$

But $\delta x = 0$ at fixed Points, i.e. at instants t_1 and t_2

$$\therefore \left[m \dot{x}\, \delta x\right]_{t_1}^{t_2} = 0.$$

The equ^n (2) gives

$$-\int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x})\, \delta x\, dt - \int_{t_2}^{t_1} K x\, \delta x\, dt = 0$$

i.e $$\int_{t_1}^{t_2} (m \ddot{x} + K x)\, \delta x\, dt = 0.$$

As δx is arbitrary, The above equ^n is satisfied only if

$$m \ddot{x} + K x = 0$$

which is The equ^n motion for one-dimentional harmonic oscillator.

Derivation of Euler's Lagrange's equation from Hamilton's Principle

The Lagrangian function is given by

$$L = L(q_1, q_2, q_3, \ldots q_k, \dot{q}_1, \dot{q}_2, \dot{q}_3 \ldots \dot{q}_k \ldots t) \quad \text{--- (1)}$$

In brief we may write $L = L(q, \dot{q}, t)$.

If the Lagrangian does not depend upon time explicitly, we have

$$L = L(q, \dot{q}).$$

$$\therefore \delta L = \sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k$$

Integrating, we get

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k}\delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\, dt \quad \text{--- (2)}$$

But According to Hamilton's Principle.

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

$$\int_{t_1}^{t_2} \sum_k \frac{\delta L}{\partial q_k}\delta q_k\, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\, dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k}\delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\delta q_k)\, dt = 0$$

because $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k}\delta q_k\, dt + \sum_k \left[\left\{\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right\} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k\, dt\right] = 0 \quad \text{--- (3)}$$

But $\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} = 0$ because $\delta q_k = 0$ at the end points according to second condition of Hamilton's principle.

Therefore (3) becomes

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt - \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt = 0$$

$$\text{or } \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \, dt = 0 \quad \text{--- (4)}$$

But variables being independent, the variation δq_k are independent if and only if $\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$

i.e. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \; K = 1, 2, 3, \ldots$

which is Lagrange's Equation.

Modified Hamilton's Principle :-

According to the Hamilton's Principle we have

$$\delta J = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt = 0 \quad \text{--- (1)}$$

The relation between Lagrangian and Hamiltonian is given by

$$H(q, p, t) = \sum_k p_k \dot{q}_k - L(q, \dot{q}, t).$$

$$\therefore L(q, \dot{q}, t) = \sum_k p_k \dot{q}_k - H(q, p, t).$$

Then eqn (1) becomes

$$\delta J = \delta \int_{t_1}^{t_2} \left[\sum p_k \dot{q}_k - H(q, p, t)\right] dt = 0 \quad \text{--- (2)}$$